When Markets Dream in Cycles

Rational Bubbles and the Financial Accelerator

Suleman Dawood Bjarni G. Einarsson Adam Lee Robertson Wang

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Contents

Abstract

We take the model developed by Bernanke & Gertler (1989) and introduce bubbles in the style of Martin & Ventura (2012) to demonstrate that within an economy with financial frictions, expectations on the value of unproductive assets can create cyclical and persistent effects in the real economy in the absence of productivity shocks. Our simple modification allows us to show that changes in entrepreneurial balance sheets can be driven purely by shifts in investor sentiments within the market for bubbles. Bubbles can be expansionary and Pareto improving so long as they continue to exist. But for bubbles to continually exist, the expectation that they will be purchased must be maintained throughout an infinity of generations. This fragility is what causes the cyclical nature of output. As these bubbles arise and disappear, balance sheet considerations can initiate and propagate changes in capital formation and output through the financial accelerator.

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"The market literally lives on its own dreams,
and each individual at every moment of time
is perfectly rational to be doing what he is doing."
Paul A. Samuelson (1957)

1 Introduction and Empirical Observations

In recent financial history, outsized asset price movements have become an empirical regularity in developed financial markets. Moreover, the real economy booms when asset prices rise, and contracts when asset prices fall. Often, movements in asset prices seemingly occur without reference to changes in productivity or technology. One of the hypothesised explanations for this phenomenon is the presence of frictions in financial markets. Such frictions can give rise to a financial accelerator, the mechanism of which works as follows. The efficient workings of the economy are constrained by the financial friction so that small shocks to the real economy are amplified and propagated by the inability of financial markets to fully intermediate funds. We seek to model fluctuations in real economic variables as the result of movements in asset prices. These movements are driven purely by investor sentiment and without reference to changes in the real economy. In this project, we are interested in whether expectations underlying rational bubbles can induce cycles in aggregate output and capital formation.

We use the framework developed in Bernanke & Gertler (1989) to model an economy with a financial friction. The friction comes in the form of a costly state verification problem, in which lenders must incur a cost to verify the outcome of a borrower's investment project while borrowers can freely observe their project outcome. We make the simple modification of allowing agents to create and sell unproductive assets, or "bubbles", the value of which they can then use as collateral when borrowing. This allows us to capture expectational shifts on the size of unproductive assets. We model these unproductive assets as rational bubbles with the same conditions as introduced in Tirole (1985), with the stochastic elements as in Martin & Ventura (2012). Changes in expectations will alter asset prices and will impact real economic activity through the following mechanism. A fall in asset prices results in a fall in the net worth of agents. This will impede agent's ability to borrow, leading to a fall in the stock of capital and a decrease in the wages earnt in the following period which will impede the ability of next period's agent to borrow. In this way, the rise and fall in capital formation and output are accelerated.

Rational bubbles are purely unproductive assets which do not bind their issuer to a payment. Rather, they derive their value from the expectation that they will be purchased by other buyers. In the current setup, rational bubbles will reduce the agency costs associated with borrowing from lenders. The creation and sale of bubbles is an additional way that borrowers can raise funds. In this way, rational bubbles can be expansionary to the extent that they allow entrepreneurs to reduce their exposure to the financial market friction. In turn, changes in the expectations of the size of the bubble will change the im-

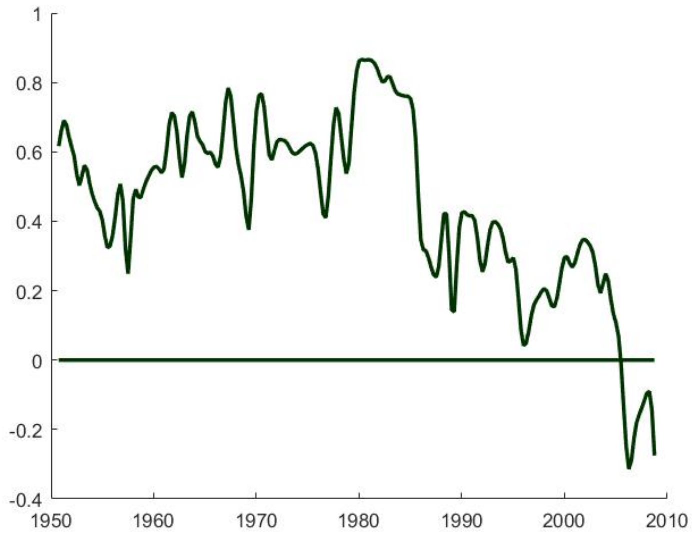

Figure 1: Correlation of productivity with output.

Band-pass filtered data. 6 year rolling window.
Data taken from Galí & van Rens (2010).

pact of the financial friction. This can cause fluctuations in capital and output as bubbles materialise or collapse in response to these shifting expectations.

We are interested in this question because of the observation presented in figure 1. The correlation between productivity and aggregate output growth has steadily fallen since the middle of the 1980's. Using reasonable calibrations for parameters, models developed in the real business cycle literature have trouble utilizing total factor productivity shocks to explain the large fluctuations in aggregate output.

Furthermore, it has been empirically observed that there exists a significant correlation between output growth and investment, and that this correlation is higher than that seen for non-durable consumption or hours worked.

Table 1: St. Dev and Correlation of selected macro-variables with GNP

Variable	St. Dev (%)	Correlation with GNP
GNP	1.72	1
Consumption (non-durables)	0.86	0.77
Investment (gross private domestic)	8.24	0.91
Hours Worked	1.59	0.86

Data taken from Cooley & Prescott (1995). Series have been de-trended using the Hodrick-Prescott filter and cover the United States over the period from 1954 to 1991.

The paper proceeds as follows. In section 2, we review the literature on the financial accelerator and that on rational bubbles. Section 3 develops the original Bernanke & Gertler model, amending it in a simple fashion to allow for bubble issuance. Section 4 presents a simulation of the model and section 5 discusses possible interpretations of the results. Section 6 concludes.

2 Literature Review

2.1 Financial Frictions Literature

The literature on financial accelerators largely stem from two seminal papers Bernanke & Gertler (1989) and Kiyotaki & Moore (1997). The term financial accelerator was originally coined in Bernanke et al. (1996). In Bernanke & Gertler (1989), the friction in financial markets is in the form of a costly state verification problem for lenders so that a financial accelerator effect arises wherein changes in balance sheets, driven by productivity shocks, initiate and propagate cyclical fluctuations in output. Carlstrom & Fuerst (1997) extend the analysis to a dynamic model with infinitely lived agents. Kiyotaki & Moore model a limited collateral pledgeability problem on the part of borrowers so that a financial accelerator arise wherein changes in asset prices drive changes in net worth which further drive changes in asset prices. The underlying commonality of models of the financial accelerator is to show that financial market conditions can amplify economic shocks. In these models, a clear link between events in the real economy and events in financial markets. For a comprehensive survey of the theoretical and empirical literature, see Brunnermeier et al. (2012) or Quadrini (2011). For a survey with a narrower focus on entrepreneurial decision making under financial frictions, see Buera et al. (2015).

Most models with financial frictions analyse dynamics by shocking real variables. In these models, the purpose of the financial friction is to amplify and propagate the small real shock by affecting the feasibility of production decisions. For an example of this, see Quadrini (2011).

2.2 Rational Bubbles Literature

The literature on rational bubbles began with three papers: Samuelson (1958), Hahn (1966), and Shell & Stiglitz (1967). The first two papers make arguments for the feasibility of financial bubbles in economies populated by rational agents. The last paper introduces behavioural explanations, such as adaptive expectations, that allow for the existence of bubbles. Samuelson showed that social contracts can lead to intergenerational transfers which result in Pareto improving outcomes within an economy with an infinite horizon of overlapping generations. Tirole (1985) extends this argument by showing that in the presence of dynamic inefficiency, rational bubbles can serve as a store of value by which productive investment can be returned to its efficient level. More generally, this strain of the rational bubbles literature emphasizes the positive effects that bubbles can have on the economy.

Tirole (1985) develops rigorous conditions on economies that can support rational bubbles, including the constraints that the bubble cannot offer a rate of return greater than the real interest rate and that the growth rate of the bubble must be bounded by the growth rate of the economy. The underlying argument contained in these models (which is the asymptotic argument made rigorous by Tirole), is that within a setting featuring infinite time without a representative agent, rational bubbles cannot be ruled out by a standard backwards induction argument. This follows since there is no terminal time period, and so it can be rational to expect a buyer to appear next period, preventing the previous owner from being left with a worthless asset.

The recent financial crisis of 2008 has spurred a new strain of literature on rational bubbles. Demarzo et al. (2008) develop a model where relative wealth effects induce market participants to participate in bubble schemes as long as others continue to do so. Scherbina (2008) develops a model and provides empirical evidence supporting the idea that bubbles are a result of analyst's advising the market to buy equities despite having private beliefs about the value of the equities that would not justify purchasing them.

Martin & Ventura (2011, 2012) develop models that emphasize the role of rational bubbles in alleviating market inefficiencies. In the former, Martin & Ventura show that a bubble, represented as an additive term on the net value of a firm, can relax collateral constraints and move the economy closer to the first-best case. They then show that the effects of the bubble shock are markedly different than a shock to productivity. In the latter they present conditions under which stochastic bubbles can exist in equilibrium within an economy featuring a financial friction. Further, they show that rational bubbles can be expansionary in the presence of a financial friction.

3 The Model

3.1 Setup

The baseline for the model is the framework established in Bernanke & Gertler (1989). The model features a discrete time overlapping generations model in which agents live for two periods. Each generation has a population normalized to 1 which consists of two types of agents: entrepreneurs with mass η and non-entrepreneurs with mass $1-\eta$. Throughout, we follow the notation established in Bernanke & Gertler: variables which refer to entrepreneurs will be superscripted. Those which refer to non-entrepreneurs will be unadorned.

Both entrepreneurs and non-entrepreneurs are endowed with a fixed amount of labour to supply in the first period of their life. The total labour endowment per generation sums to 1: $\eta L^e + (1-\eta)L = 1$.

Agents have risk-neutral preferences and only care about old age consumption.[1] In

[1] This is a slight simplification of the model in Bernanke & Gertler (1989), who assume that lenders consume in both periods.

particular, agents have preferences $\mathbb{E}_t[c_{t+1}]$. We denote the market wage by w_t. The savings of entrepreneurs are therefore:

$$S_t^e = w_t L^e. \tag{1}$$

We assume that entrepreneurs and non-entrepreneurs are endowed with the same amount of labour, which we now normalise to 1. The savings of non-entrepreneurs is given by:[2]

$$S_t = w_t L \tag{2}$$

There are two goods in this economy: a capital good (k) and an output good (y). Output can either be consumed in the current period t, or moved to period $t+1$, through an investment or storage technology. Capital depreciates fully each period and there is no population growth. The output good is produced via a production function, which yields per capita output as a function of capital per capita, where $\theta > 0$ is a non-stochastic term representing total factor productivity:

$$y_t = \theta f(k_t). \tag{3}$$

Output can be stored in the risk-less storage technology, which offers a gross rate of return of $r \geq 1$ of the output good.

Because of the assumption of full depreciation, for $k_t > 0$ it must be the case that a proportion of output in $t-1$ was used in the investment technology which is described as follows. The investment technology consists of indivisible projects, one for each entrepreneur and which cannot be transferred. Entrepreneurs themselves have a productivity type, $\omega \sim U[0,1]$, which determines the cost of their project. Costs are in terms of the output good and are given by the function $x(\omega)$ which is increasing in ω. Projects that are undertaken produce a quantity of capital available for use in the next period. The outcome of projects is a discrete random variable, with two possible values $k_L < k_H$, and associated probabilities π_L, π_H.[3] The expected outcome is κ. We assume $\kappa > \frac{r}{\hat{q}}$, where $\hat{q}$ is the expected price of capital (in terms of the output good) in the following period.

The outcome of any project can be costlessly observed by the entrepreneur conducting it. For other agents to observe the outcome, they must pay an auditing cost of $\gamma > 0$ in terms of the capital good. As other agents cannot costlessly observe the outcome, the entrepreneur who opens the project is able to lie about the outcome of their project. We assume that lenders can pre-commit to auditing with some probability, which may depend on the announced outcome. We also assume that the outcome of concurrent investment projects are mutually independent.

[2] Bernanke & Gertler make the assumption that the mass of savings is strictly greater than the total amount of worthwhile projects so that storage is always used in equilibrium and r is the prevailing interest rate.

[3] Bernanke & Gertler (1989) consider a more general model, with n possible outcomes.

We make the following assumptions on the production function:

$$\theta f'(0)\kappa > rx(0) + \gamma \tag{4}$$

$$\theta f'(\kappa\eta) < rx(1), \tag{5}$$

This implies that the most productive entrepreneurs will always invest even if audited with certainty and the least productive investor will never invest. That is, some but not all entrepreneurs will always want to invest.

3.2 Introducing Bubbles

We modify the model in Bernanke & Gertler (1989) by allowing entrepreneurs to issue bubbles. Bubbles are costless for entrepreneurs to create and are not productive, so that when they are first created and sold entrepreneurs receive a payment for nothing in return. Agents may choose to purchase bubbles because they believe they can sell them on to other agents in the future. In this way, bubbles are similar to Ponzi schemes: participants that place money into the scheme today (buy bubbles) will obtain a return due to the participation of those in the next period (those who buy bubbles next period). A key feature of bubbles, especially pertinent in models of financial frictions, is that they do not constitute a promise by the seller to make future payments. Rather, it is the anticipation of future buyers that gives bubbles their value. In this way, when buyers buy a bubble they are not taking on the credit risk of the seller. Rather, the risk they incur is whether or not the young of the next period will purchase their bubbles.

The value of the total stock of bubbles in any given period is the sum of the value of new bubbles issued plus the value of those issued in previous periods. That is $b_t = b_{t-1} + b_t^n$. There are two conditions which must be satisfied for bubbles to exist. First, the bubble must offer a rate of return in expectation that is sufficient to induce agents to purchase it. For simplicity, we will focus on the case where the bubbles are expected to return the same rate as the storage technology:

$$\frac{\mathbb{E}_t(b_{t+1})}{b_t} = r. \tag{6}$$

If bubbles were expected to return less than this rate, then lenders would prefer to use the risk-less storage technology and so a positive stock of bubbles could not be a part of equilibrium. It may be possible for equilibria to exist in which bubbles are expected to return more than r but certainly not more than $\hat{q}\kappa$. If bubbles returned more than $\hat{q}\kappa$ no one would want to invest in entrepreneurial projects, which would result in a null capital stock, and therefore no wages with which to purchase next period's bubbles. A one-period backward induction argument then rules out the existence of bubbles in this case.

At the rate of return defined by (6), lenders will be indifferent between lending and purchasing bubbles. To ensure that markets will clear, there must be enough funds in the economy to cover the total costs of investment in equilibrium, as well as the purchase of all new and existing bubbles:

$$(1 - \eta)w_t \geq c(\cdot) - \eta w_t + b_t, \tag{7}$$

We let the $c(\cdot)$ function represent the total cost of investment in equilibrium, which will depend on the amount of new bubbles issued; we will return to this constraint in subsection 3.5 below, after we have developed the machinery required to characterise $c(\cdot)$.

If (7) did not hold in each period, then a backward induction argument would rule out the existence of bubbles in equilibrium.[4] This condition can also be interpreted as a limit on the size of bubbles. Furthermore, given the assumption that $\kappa > \frac{r}{\hat{q}}$, standard functional form specifications on the production function, and making appropriate assumptions on parameter levels the economy and thus wages will always grow at a rate greater than bubble growth so that the above expected growth rate of bubbles can be maintained in equilibrium.

Of importance here is how much of the stock of new bubbles each entrepreneur sells. Notationally, we will write $\int_0^1 b_t^n(\omega)\,\mathrm{d}\omega = b_t^n$, where $\int_{\omega_1}^{\omega_2} b_t^n(\omega)\,\mathrm{d}\omega$ is the amount of bubbles that individuals between $\omega_1 < \omega_2$ sell. For the sake of simplicity, we assume each entrepreneur sells the same amount of bubbles. That is, $b_t^n(\omega) = f(\omega)b_t^n = b_t^n$ where $f(\omega) = 1$ is the $U[0,1]$ pdf.

Now, we define entrepreneur's net wealth as the sum of their wages and the income from the bubbles they have sold, so that total entrepreneurial net wealth is $\eta w_t + b_t^n$.[5] With this in mind, we move onto solving for the optimal financial contract that entrepreneurs create with lenders.

3.3 The Optimal Financial Contract

The derivation of the optimal financial contract is very close to that of Bernanke & Gertler (1989) with the role of entrepreneurial savings replaced with their net wealth, which is inclusive of their sales of bubbles. The problem can be stated as follows. Let p be the probability of an audit in the bad state, c_i be the consumption of the entrepreneur when he declares outcome $i \in \{H, L\}$, and let c^a be his consumption when he announces the

[4]In the period prior to the bubbles being too expensive for the economy to afford, agents would not wish to buy them. Therefore, agents in the previous period would not wish to buy them and so on.

[5]Since any entrepreneur is indifferent between using the storage technology and buying bubbles, we assume that when faced with this choice, they use the storage technology.

“bad state” L and gets audited. The optimal financial contract solves

$$\begin{aligned}
\max_{\{p,c_L,c_H,c^a\}} \pi_L(pc^a + (1-p)c_L) + \pi_H c_H \quad \text{subject to:} \\
\pi_L[\hat{q}k_L - p(c^a + \hat{q}\gamma) - (1-p)c_L] + \pi_H[\hat{q}k_H - c_H] \geq r(x(\omega) - NW^e) \\
(1-p)(\hat{q}(k_H - k_L) + c_L) \leq c_H \\
c_L \geq 0 \\
c^a \geq 0 \\
0 \leq p \leq 1
\end{aligned} \tag{8}$$

where $\hat{q}$ is the expected relative price of capital. Just as in Bernanke & Gertler (1989), we assume that agents take $\hat{q}$ as parametric.

The solution to this problem falls into two cases. If the net worth of an entrepreneur is large enough such that he can pay back the lender the required return even in the low state, i.e.

$$\hat{q}k_L \geq r(x(\omega) - NW^e) \tag{9}$$

there is no agency problem and the optimal auditing probabilities are zero. As in Bernanke & Gertler (1989), we will refer to this as the “full collateralisation” case. The expected consumption of the entrepreneur in this case is

$$\hat{c}_{fc} = \hat{q}\kappa - r(x(\omega) - NW^e), \tag{10}$$

where we note that the latter term $r(x(\omega) - NW^e)$ can be either positive or negative: if bubbles are large enough that the entrepreneur does not need to borrow funds from lenders, he will earn a return of r on the excess, $NW^e - x(\omega)$.

The second, and more interesting case is when entrepreneur’s net worth is small enough relative to the project costs such that he would be unable to meet the required repayment in the bad state. Here, we are in the “incomplete collateralisation” case. The optimal auditing probability when the low outcome is announced is

$$p = \frac{r(x(\omega) - NW^e) - \hat{q}k_L}{\pi_H \hat{q}(k_H - k_L) - \pi_L \hat{q}\gamma}$$

The expected consumption of the entrepreneur in this case is given by

$$\hat{c}_{ic} = \psi[\hat{q}\kappa - r(x(\omega) - NW^e) - \pi_L \hat{q}\gamma]$$

where $\psi = [\pi_H \hat{q}(k_H - k_L)]/[\pi_H \hat{q}(k_H - k_L) - \pi_L \hat{q}\gamma] > 1$.[6]

[6] See appendix for the derivation of these results.

3.4 The Investment Decision

As in Bernanke & Gertler (1989) we can split our measure of entrepreneurs into 3 types. We implicitly define $\underline{\omega}$ and $\overline{\omega}$ as follows:

$$\hat{q}\kappa - \hat{q}\pi_L\gamma = rx(\underline{\omega}) \tag{11}$$

$$rx(\overline{\omega}) = \hat{q}\kappa, \tag{12}$$

Now, we can identify our three groups:

(i) Entrepreneurs who will always invest (even if audited with probability 1): those who draw $\omega \in [0, \underline{\omega}]$;

(ii) Entrepreneurs who *may* invest, depending on their net wealth: those who draw $\omega \in (\underline{\omega}, \overline{\omega})$;

(iii) Entrepreneurs who will never invest, even if auditing never occurs: those who draw $\omega \in [\overline{\omega}, 1]$.

As in Bernanke & Gertler (1989) we allow entrepreneurs in group (ii) to take part in a lottery, given as follows. Define $NW^*(\omega)$ to be the quantity such that (9) is satisfied with equality:

$$NW^*(\omega) = x(\omega) - \frac{\hat{q}}{r}k_L. \tag{13}$$

Then, any entrepreneurs in group (ii) with net wealth below $NW^*(\omega)$ will be willing to engage in the following lottery:

$$\text{Lottery payoff } = \begin{cases} NW^*(\omega) & \text{with probability } \quad \frac{NW^e}{NW^*(\omega)} \\ 0 & \text{otherwise} \end{cases}. \tag{14}$$

By participating in this lottery, these entrepreneurs expect to obtain the fully collateralised level of expected consumption $\hat{c}_{fc}$ with probability $\frac{NW^e}{NW^*(\omega)}$ and 0 otherwise. In the appendix, we prove that any such group (ii) entrepreneur would prefer to take part in this lottery than either investing all their funds in the investment technology or exclusively using the storage technology. Writing c^L to mean the expected consumption from participating in this lottery, we have that $c^L > \hat{c}_{ic}$ and $c^L > rNW^e$. Therefore, it also follows that such entrepreneurs would prefer to participate in this lottery with the entirety of their net wealth.

3.5 Law of Motion for Capital

Since we have assumed full depreciation, the capital stock will be inherited in each period by the investment projects opened in the prior period. The capital stock directly enters the production function and because of the specification that preferences are only over consumption, labour is supplied inelastically therefore output, wages and savings are fully determined by the capital stock in each period.

We first consider the perfect information case, wherein lenders can costlessly observe the outcomes of investment projects. Here, bubbles cannot exist as they will violate the resource constraint (7). Under perfect information, the capital production is at full capacity as all entrepreneurs who can open projects that return at least as much as the storage technology are fully funded. An expectation of bubble growth would require wages to grow at least at the same rate as the bubble. However, this would require an ever greater amount of capital to enter the production function in every period which is not possible given that the economy is already operating at full capital production. Therefore, the analysis of this case is exactly the same as in Bernanke & Gertler (1989). Savings of the young will go to either entrepreneurs or storage. The last entrepreneur to invest in this case will be such that the opportunity cost of investing is exactly equal to the benefit of investing. As $\omega \sim U[0,1]$, by definition $\overline{w}$ (see (12)) is exactly the fraction of entrepreneurs which will meet the risk-free investment hurdle. Let i_t be the number of investment projects undertaken,

$$i_t = \eta \overline{w} \tag{15}$$

Which implies that the law of motion of capital is given by:

$$k_{t+1} = \kappa \eta \overline{w}. \tag{16}$$

Now we consider the imperfect information case in which lenders must pay γ in order to observe investment outcomes. For all entrepreneurs with productivity $\omega < \underline{\omega}$ we have that the optimal probability of audit is given by:

$$p(\omega) = \max\left(\frac{rx(\omega) - \hat{q}k_L - rNW^e}{\hat{q}(\pi_H(k_H - k_L) - \pi_L\gamma)}, 0\right). \tag{17}$$

For those entrepreneurs with $NW^e \geq NW^*(\omega)$, $p(\omega) = 0$, where (13) defines $NW^*(\omega)$. For all other entrepreneurs of this type, bubbles affect the optimal auditing probability negatively through NW^e; the optimal probability is also decreasing in $\hat{q}$. For all entrepreneurs with type $\omega \in (\underline{\omega}, \overline{\omega})$ the specification of risk-neutral preferences implies that these entrepreneurs prefer to take part in the collateralisation lottery discussed in the previous section.

Therefore, entrepreneurs of type (ii) will not face agency costs when they invest but only a fraction of them will be able to invest. As in Bernanke & Gertler (1989), define $g(\omega)$ for this type as:

$$g(\omega) = \min\left\{\frac{NW^e}{NW^*(\omega)}, 1\right\}. \tag{18}$$

Note that provided $NW^e < NW^*(\omega)$, $g(\omega) < 1$ as for $p(\omega)$, bubbles affect this positive thorough NW^e, and $g(\omega)$ is also increasing in $\hat{q}$.

Total capital formation per capita is given by:

$$k_{t+1} = \left(\kappa\underline{\omega} - \int_0^{\underline{\omega}} \pi_L \gamma p(\omega)\, \mathrm{d}\omega + \int_{\underline{\omega}}^{\overline{\omega}} \kappa g(\omega)\, \mathrm{d}\omega\right)\eta \tag{19}$$

The term $\left(\kappa\hat{\omega} - \int_0^{\underline{\omega}} \pi_1 \gamma p(\omega) d\omega\right)\eta$ captures the capital formation stemming from investment by entrepreneurs of type (i) who have productivity $\omega \leq \underline{\omega}$ and $\left(\int_{\underline{\omega}}^{\overline{\omega}} \kappa g(\omega) d\omega\right)\eta$ captures the capital formation of by type (ii) entrepreneurs with productivity $\omega \in (\underline{\omega}, \overline{\omega})$ who win the lottery and so invest with full collateralisation.

In the case where investors of type (i) are not fully collateralised (and so face a positive probability of auditing) and that those of type (ii) who take part in the lottery do not all win we can re-express (19) as follows, being explicit about the terms inside the integral:

$$k_{t+1} = \left(\kappa\underline{\omega} - \int_0^{\underline{\omega}} \pi_L \gamma \frac{rx(\omega) - \hat{q}k_L - r(S^e + b_t^n)}{\hat{q}(\pi_H(k_H - k_L) - \pi_L\gamma)}\, \mathrm{d}\omega + \int_{\underline{\omega}}^{\overline{\omega}} \kappa \frac{r(S^e + b_t^n)}{rx(\omega) - \hat{q}k_L}\, \mathrm{d}\omega\right)\eta \tag{20}$$

If we were to set $b_t^n = 0$ in the above, the law of motion would be the same as in the original Bernanke & Gertler model.

We now return to the feasibility condition on bubbles, (7), and characterise it more explicitly. Using the results developed above, $c(\cdot) = \int_0^{\underline{\omega}} x(\omega) + \pi_L\gamma p(\omega)\, \mathrm{d}\omega + \int_{\underline{\omega}}^{\overline{\omega}} x(\omega)g(\omega)\, \mathrm{d}\omega$, and so (7) is:

$$(1-\eta)w_t \geq \int_0^{\underline{\omega}} x(\omega) + \pi_L\gamma p(\omega)\, \mathrm{d}\omega + \int_{\underline{\omega}}^{\overline{\omega}} x(\omega)g(\omega)\, \mathrm{d}\omega - \eta w_t + b_t, \tag{21}$$

Now we turn to the question of the size of b_t^n. It is costless for entrepreneurs to create bubbles, and in doing so they increase their income, which can either be put towards their investment project or stored to obtain the rate of return r. This increases the entrepreneur's expected consumption in the following period (both $\hat{c}_{fc}$, c^l, $\hat{c}_{ic}$ are increasing in NW^e) and so they will create and sell as many bubbles as they can in each period. That is, we will have (21) binding:

$$b_t^n = w_t - \int_0^{\underline{\omega}} x(\omega) + \pi_L\gamma p(\omega)\, \mathrm{d}\omega - \int_{\underline{\omega}}^{\overline{\omega}} x(\omega)g(\omega)\, \mathrm{d}\omega - b_{t-1}. \tag{22}$$

We note here that new bubbles enter the right-hand side of this equation in two places. As bubble issuance brings expected auditing costs down, there is a positive effect through $p(\omega)$. The effect through $g(\omega)$ is negative, which is due to a greater proportion of investors winning the collateralisation lottery and therefore avoiding the agency cost.

Once more assuming that investors of type (i) are not fully collateralised, and that those of type (ii) do not all win, we can find an explicit form for new bubble issuance, since

(22) takes the form:

$$b_t^n = w_t - \int_0^{\underline{\omega}} x(\omega)\,\mathrm{d}\omega - \pi_L\gamma \int_0^{\underline{\omega}} \frac{rx(\omega) - \hat{q}k_L - r(S^e + b_t^n)}{\hat{q}(\pi_H(k_H - k_L) - \pi_L\gamma)}\,\mathrm{d}\omega$$
$$- \int_{\underline{\omega}}^{\overline{\omega}} x(\omega)\frac{S^e + b_t^n}{x(w) - \hat{q}k_L r^{-1}}\,\mathrm{d}\omega - b_{t-1}. \quad (23)$$

Rearranging this, we obtain:

$$b_t^n = \frac{w_t - \int_0^{\underline{\omega}} x(\omega)\,\mathrm{d}\omega - \int_0^{\underline{\omega}} \frac{rx(\omega) - \hat{q}k_L - r(S^e)}{\hat{q}(\pi_H(k_H - k_L) - \pi_L\gamma)}\,\mathrm{d}\omega - S^e \int_{\underline{\omega}}^{\overline{\omega}} \frac{x(\omega)}{x(\omega)\hat{q}k_L r^{-1}}\,\mathrm{d}\omega - b_{t-1}}{1 - \frac{r\pi_L\gamma\underline{\omega}}{\hat{q}(\pi_H(k_H - k_L) - \pi_L\gamma)} + \int_{\underline{\omega}}^{\overline{\omega}} \frac{x(\omega)}{x(\omega) - \hat{q}k_L r^{-1}}\,\mathrm{d}\omega}. \quad (24)$$

4 Model Simulation

In order to illustrate the workings of the model we simulate it numerically for the case that applies immediately above. To run the simulation we need to make functional assumptions for the production function and the cost of investment for entrepreneurs. For the production function we will assume the following CES form:

$$F(K_t, L_t) = [(1-\alpha)L_t^{\frac{\sigma-1}{\sigma}} + \alpha K_t^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}.$$

For entrepreneur's cost of investment we assume the following functional form:[7]

$$x(\omega) = (c + \omega)^\delta.$$

The model economy can be in two states, a "fundamental" or "no bubble" state (F) in which agents believe the bubble will not be re-sellable next period and are thus not willing to purchase it in the current period leading to a collapse of the bubble, and a "bubbly" state (B) in which agents do believe the bubble will be re-sellable in the following period. For simplicity, and following Martin & Ventura (2011), we assume the existence of a sunspot variable, $\Lambda_t \in \{F, B\}$, that determines the state of the economy. Let Λ_t follow a Markov process where λ_H is the probability of being in the bubbly state, conditional on having been in the bubbly state last period and λ_L be the probability of being in the bad state conditional on having been in the bad state in the last period. Finally, we need to assume some parameter values. These are presented in table 2.

[7]This is increasing in ω as required, provided that $\delta(c+\omega)^{\delta-1} > 0$, which is satisfied by the parametrisation used here.

Table 2: Parametric assumptions for simulation

Variable	Description	Value
α	Relative productivity of capital	0.33
σ	Elasticity of substitution in production	1.5
θ	Total factor productivity	1.2
η	Fraction of entrepreneurs	0.25
c	Cost function parameter	1
δ	Cost function parameter	2
k_L	Low investment outcome	0.5
k_H	High investment outcome	1
π_L	Prob. of low outcome	0.25
π_H	Prob. of high outcome	$1-\pi_L$
λ_L	Prob. of remaining in fundamental state	0.4
λ_H	Prob. of remaining in bubbly state	0.8
r	Return on storage	1.02
γ	Auditing cost	$0.5k_L$

Figure 2 plots simulation results for the capital stock and the bubble for the cases of perfect information and asymmetric information. As discussed above, under perfect information bubbles cannot exist and therefore cannot cause fluctuations in the capital stock. Under asymmetric information however, we see that periods where the bubble bursts are associated with a fall in the capital stock, as the model predicts (see (20)). Figure 3 shows the optimal auditing probability $p(\omega)$ with and without a bubble and illustrates the fact that the introduction of a bubble to the economy lowers the optimal auditing probability.

Figure 4 graphs impulse response functions for the capital stock, the stock of bubbles, the fraction of good entrepreneurs ($\underline{\omega}_t$), and the fraction of projects with full collateralisation for a scenario in which the bubble bursts in period 5. By design, the stock of bubbles evaporates with an associated fall in the capital stock. In following period the economy returns to the bubbly state and both variables return to their previous values.

The fall in the capital stock raises the marginal product of capital and thus increases $\hat{q}_t$ which results in a rise in $\underline{\omega}_t$, the fraction of good entrepreneurs, which lasts for one period before reverting to its previous value. Intuitively this is because the higher expected price of capital makes it attractive for higher cost types to invest. The fraction of projects with full collateralisation, however, falls in tandem with the bubble bursting. In the following period, because $b_{t-1} = 0$ in equation (22), the fraction rises substantially as bubble issuance resumes and the current cohort of entrepreneurs are able to issue the entirety of the stock of bubbles and thus have an unusually high net worth. Two periods after the bubble collapses the fraction of projects with full collateralisation goes back to its pre-crash level.

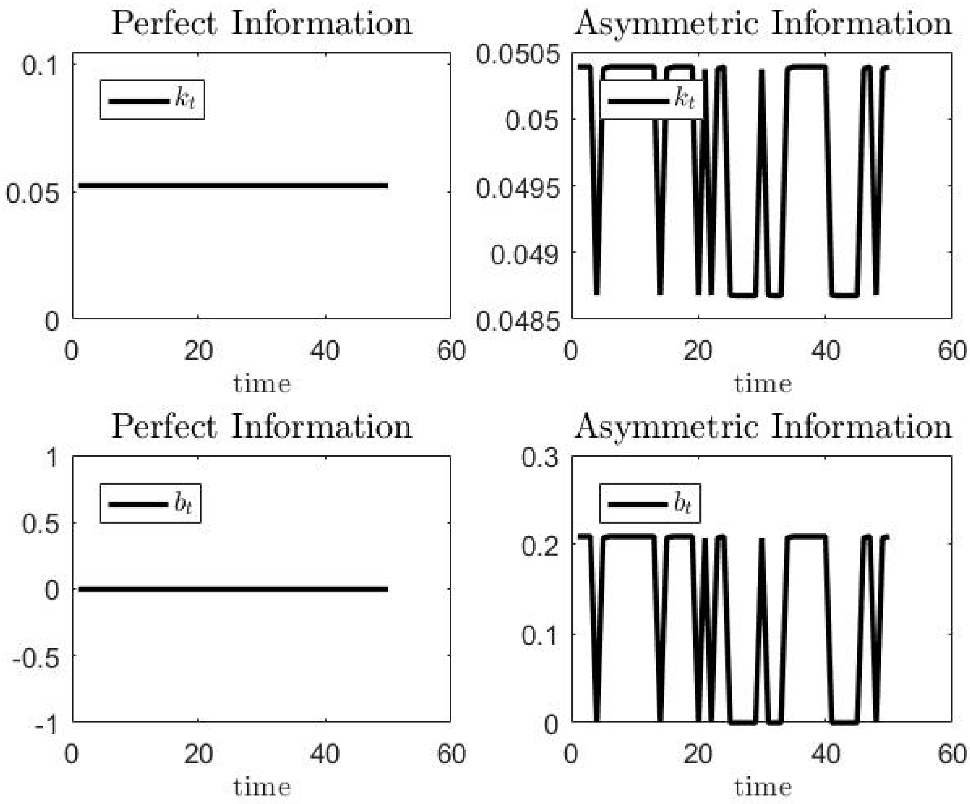

Figure 2: Simulations from the model with perfect and asymmetric information.

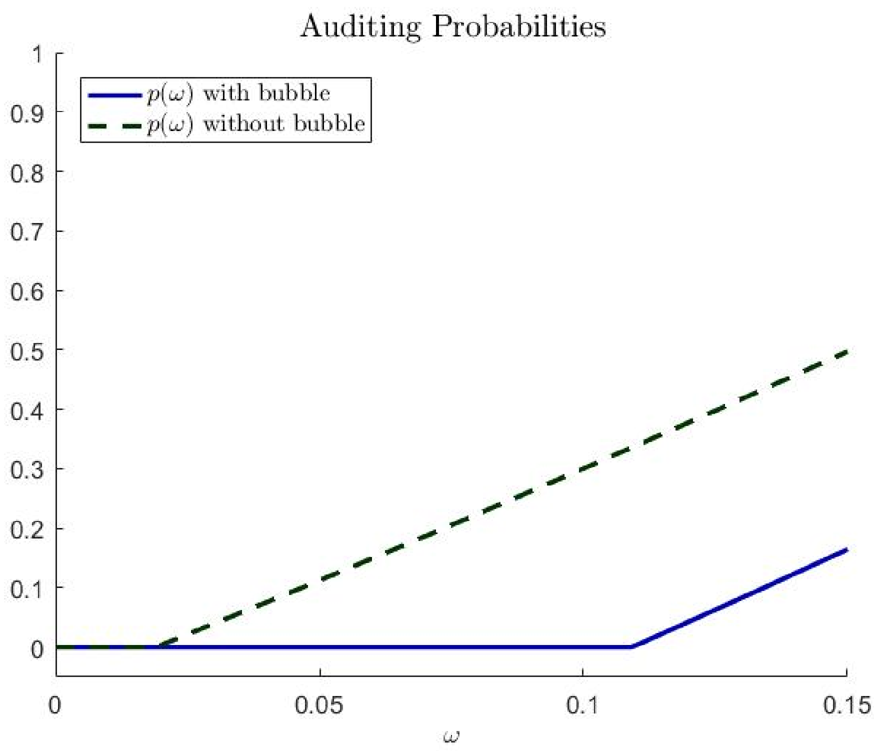

Figure 3: Auditing probabilities with and without a bubble based on model simulation

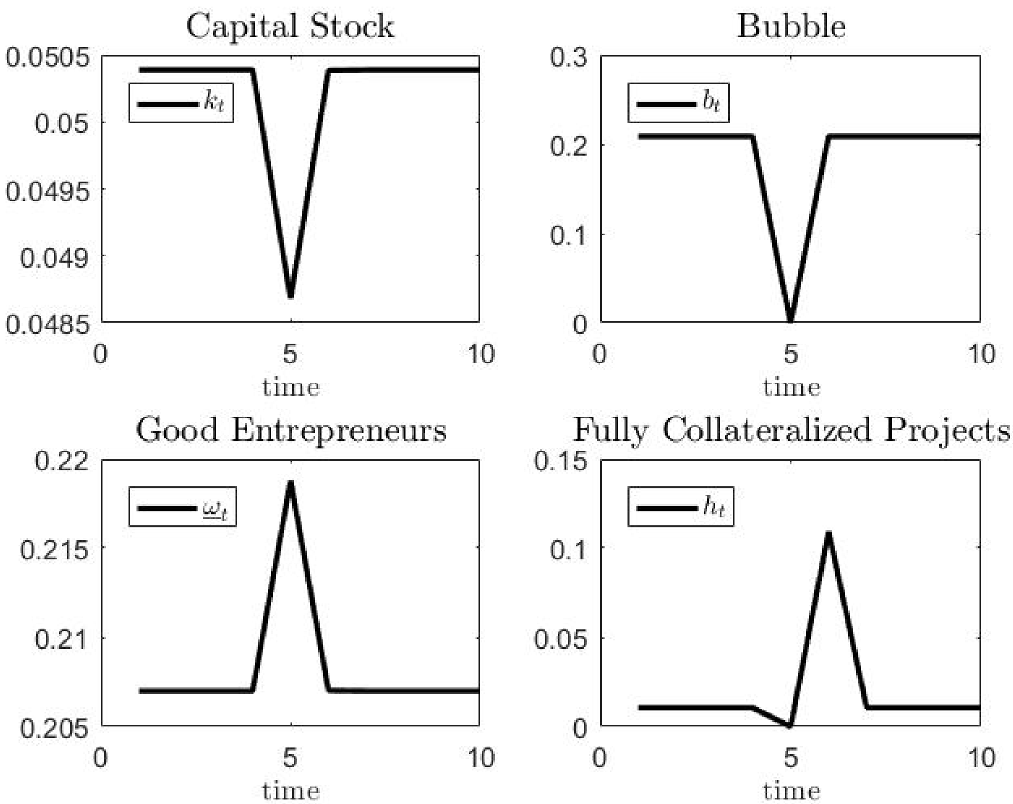

Figure 4: Impulse-response to a collapse in the stock of bubbles

5 Discussion

The model we have presented is a simple modification of the model presented by Bernanke & Gertler (1989). We have replaced the stochastic TFP with a constant level of productivity. We allow entrepreneurs to issue rational bubbles, which enter the optimization as an additive term in their net wealth and in this way alleviates the financial friction. Further, expectations on the bubble vary stochastically. Almost all the mathematics and, in particular, the transmission mechanisms embedded in our model are identical to those in the original model. Nonetheless, this small modification has allowed us to show that expectations arising in financial markets out of investor sentiment, and which have no basis in shifts of real variables, can generate persistent responses in the real economy.

To understand this result, it is crucial to recognize the role that bubble creation plays in reducing the need for entrepreneurs to borrow. Agents buy bubbles because they expect future agents to also buy bubbles, not because they expect the creator of the bubble to make payments to the holders of the bubble. Consider equation (22). On one hand, bubble creation will result in a greater proportion of individuals who win the collateralization lottery. However, both bubble creation and total investment enter directly into the resource constraint, creating a natural tension in the allocation of aggregate savings. On the other hand, productive enough entrepreneurs can sell new bubbles to other young agents and use the proceeds to invest, thereby avoiding the need to borrow. This mitigates the effects of the financial friction and will result in a larger amount of capital formation and aggregate investment. This ultimately increases the income of the next period's young. Furthermore, the ability to issue bubbles hinges on whether people can rationally expect the bubble to grow. This participation is guaranteed because as the wages of the next periods young grows it is rational to expect that future generations will also purchase new and existing bubbles so that the aggregate bubble size grows.

Just as in Martin & Ventura (2012), we allow for stochastic equilibria with bubble creation in the presence of a financial friction. [8] However, rather than assuming a reduced form for investment efficiency, we have used the framework in Bernanke & Gertler (1989) to model the friction from the ground up. This allows us to show how expectations on bubbles interact with output fluctuations. We can capture the impact of the financial friction within our model by the term

$$\phi = \eta\left(\kappa[\overline{w} - \underline{w}] + \int_0^{\underline{w}} \pi_L \gamma p(\omega)\,\mathrm{d}\omega - \int_{\underline{w}}^{\overline{w}} \kappa g(\omega)\,\mathrm{d}\omega\right) \tag{25}$$

In the first best case of perfect information, $g(\omega) = 1$, $p(\omega) = 0$, and $\underline{\omega} = \bar{\omega}$ which will imply that $\alpha = 0$. The capital stock and output growth are strictly decreasing in this measure. We have shown above that $g(\omega)$ is increasing in the bubble term and $p(\omega)$ is decreasing in the bubble term so that the existence of bubbles will reduce the financial friction in

[8] Although we focus on the case of $\frac{\mathbb{E}_t[b_{t+1}]}{b_t} = r$ for simplicity

the economy. In this way, the ability to issue bubbles leads to a result closer to the first best for the economy as it reduces the contractionary effect of the financial friction. This measure of financial imperfection is comparable to other reduced form parameters used in the financial friction literature such as the measure of financial development Caballero et al. (2008) and the collateral pledgeability constraint in Kiyotaki & Moore (1997).

The arrival of expectations permitting the existence of bubbles allow entrepreneurs to create and sell bubbles, which is a pure windfall for entrepreneurs and increases their net wealth. This will increase investment activity as agency costs are mitigated because amongst suitably productive entrepreneurs the probability of being audited will decrease for some and the the odds of winning the collateralization lottery will increase for others. The disappearance of expectations that permit the existence of bubbles will decrease entrepreneurial net wealth. Therefore, more investment projects will need loans in order to be initiated so that entrepreneurs will be forced to face higher agency costs. This will directly lower capital formation and therefore the wages paid in the next period which will lower the net worth of the entrepreneurs in the following period. Thus investor sentiment, which may be entirely unrelated to the real economy, may initiate, as well as propagate, cyclical fluctuations in output.

All of the above analysis is predicated on atomistic agents taking $\hat{q}$ as exogenous. Interestingly, assuming coordination between agents would allow them to consider $\hat{q}$ endogenously and could lead to a change in social welfare. In particular, relaxing this assumption will generate more complex effects of bubble issuance. Bubble issuance will result in a change in the capital stock in the following period, leading to a change in $\hat{q}$. This will result in changing levels of $\underline{\omega}$ and $\overline{\omega}$. Similarly, the proportion of investors who win the collateralization lottery will change because the implicit definition of $NW^*(\omega)$ depends on $\hat{q}$ and new bubbles enter directly into the net wealth of entrepreneurs. Note similar effects will hold for the optimal auditing probability.

An interesting aspect of this can be seen in figure 4 from the simulation. In the period following the disappearance of the bubble, the group of entrepreneurs who are in group (i), those who will invest even if they are audited with certainty, jumps. This is because the sudden disappearance of the bubble leads to a drop in the capital stock which increases the marginal return on capital and therefore $\hat{q}$. Thus, the cutoff level for this group $\underline{\omega}$, implicitly defined by (11), will increase. This indicates a first mover advantage to issuing bubbles. Entrepreneurs in group (i) will not have to enter into a collateralization lottery and will be able to enjoy a higher level of old age consumption in expectation. This aligns with the result in Samuelson (1958) and summarized in Shell (1971), wherein intergenerational social contracts lead to Pareto improving outcomes. This particular result shows that the first generation to issue bubbles will make themselves and every subsequent generation better off, thereby leading to a Pareto improvement. This aspect is what makes rational bubbles rational - they amount to an intergenerational transfer. The current period's young buy bubbles anticipating the next period's young will buy a larger bubble and so on and so forth. The only way for this expectation to be maintained in equilibrium is if rational bubbles, valuable only because of sentiment, can impact and drive the real economy.

6 Conclusion

We have taken the model developed by Bernanke & Gertler (1989) and introduced bubbles in the style of Martin & Ventura (2012) to demonstrate that within an economy with financial frictions, expectations on the value of unproductive assets can create cyclical and persistent effects in the real economy. Our simple modification allows us to show that changes in entrepreneurial balance sheets can be driven purely by shifts in investor sentiments within the market for bubbles, without any shock to productivity. Bubbles can be expansionary and Pareto improving so long as they continue to exist. But for bubbles to continually exist, the expectation that they will be purchased must be maintained throughout an infinity of generations. This fragility is what causes the cyclical nature of output. As these bubbles arise and disappear, balance sheet considerations can initiate and propagate changes in capital formation and output, through the financial accelerator.

Going back to the empirical lack of correlation between productivity and GDP since the 1980's and the high correlation between investment and output (see figure 1 and table 1), models such as this which generate fluctuations through different channels can provide alternative frameworks for thinking about fluctuations in real variables.

Further research on this topic could take several directions. We have focused on a particular equilibrium for the bubble growth rate, investigating other equilibria could also lead to interesting results such as alternative levels of capital formation or an optimal bubble size. Furthermore, alternative assumptions on the expectation of bubble growth might generate distributional effects on wealth accumulation, especially in the presence of asymmetric beliefs between entrepreneurs and lenders or if beliefs are generated using a behavioural model. Future research could also investigate the impact of altering the distribution of bubble issuance amongst entrepreneurs. The behaviour of the government and monetary authority are absent in this model. Introducing such institutions might help mitigate the coordination problem associated with the expectation on bubbles. Lastly, an interesting extension, motivated by the observed high and positive correlation between hours worked and output, lies in relaxing the assumption on the inelastic supply of labour. As bubbles materialize and collapse, wages paid in each period will rise and fall. This will generate substitution and wealth effects, thereby directly impacting the amount of labour supplied which could further generate fluctuations in output and other real variables.

A Appendix

A.1 Collateralization lottery

In this section we prove that entrepreneurs in group (ii) with productivity levels $\underline{\omega} < \omega < \overline{\omega}$ are willing to participate in a full collateralization lottery where their probability of winning is given by $\frac{NW^e}{NW^*(\omega)}$. The collateralization lottery can be interpreted as financial institutions that collectivize savings and makes loans; institutions such as credit unions, mutual insurance, and mutual savings banks. The principal feature of these institutions is that member's deposits are accepted and pooled, which are then used to make loans to members.

Claim 1. *If we define c^l and $\hat{c}_{ic}$ as follows:*

$$\hat{c}_{ic} = \psi\{\hat{q}\kappa - r(x(\omega) - NW^e) - \pi_L\hat{q}\gamma\} \tag{26}$$

$$c^L = \frac{NW^e}{NW^*(\omega)}\Big(\hat{q}\kappa - r(x(\omega) - NW^*(\omega))\Big), \tag{27}$$

with

$$\psi = \frac{\pi_H\hat{q}(k_H - k_L)}{\pi_H\hat{q}(k_H - k_L) - \pi_L\hat{q}\gamma}, \tag{28}$$

then, for any $NW^e \in [0, NW^*(\omega))$, $c^L > \hat{c}_{ic}$ for $\omega \in (\underline{\omega}, \overline{\omega})$.

Proof. Let $\epsilon > 0$ be such that $NW^e = NW^*(\omega) - \epsilon$. Using this, $\kappa = \pi_H k_H + \pi_L k_L$, and (13), we can rewrite the inequality we wish to show as.

$$\frac{NW^*(\omega) - \epsilon}{NW^*(\omega)}\hat{q}[\kappa - k_L] > \frac{\kappa - k_L}{\kappa - k_L - \pi_L\gamma}\hat{q}[\kappa - k_L - \pi_L\gamma] - \frac{\kappa - k_L}{\kappa - k_L - \pi_L\gamma}r\epsilon. \tag{29}$$

Dividing through by the positive quantity $\hat{q}(\kappa - k_L)$, we obtain:

$$1 - \frac{\epsilon}{NW^*(\omega)} > 1 - \frac{r\epsilon}{\hat{q}(\kappa - k_L - \pi_L\gamma)},$$

that is:

$$\frac{\epsilon}{NW^*(\omega)} < \frac{r\epsilon}{\hat{q}(\kappa - k_L - \pi_L\gamma)}. \tag{30}$$

Using (13) once more, we can reduce this to:

$$\frac{r\epsilon}{rx(\omega)} < \frac{r\epsilon}{\hat{q}\kappa - \hat{q}\pi_L\gamma}. \tag{31}$$

Since $x(\omega)$ is an increasing function, using (11), we can conclude that $rx(\omega) > \hat{q}\kappa - \hat{q}\pi_L\gamma$ for $\omega \in (\underline{\omega}, \overline{\omega})$ and so (31) holds. Thus, $q^L > \hat{q}_{ic}$ as claimed. □

Claim 2. *For any $NW^e \in [0, NW^*(\omega))$, $c^L \geq rNW^e$ for $\omega \in (\underline{\omega}, \overline{\omega})$.*

Proof. Note that using (13) we can write:

$$c^L = \frac{NW^e}{NW^*(\omega)}[\hat{q}\kappa - rx(\omega)] + rNW^e. \tag{32}$$

For $\omega \in (\underline{\omega}, \overline{\omega})$, since $x(\omega)$ is increasing, by (12), $\hat{q}\kappa - rx(\omega) > 0$ and so the above equation reveals that $c^L > rNW^e$. □

A.2 Deriving the optimal financial contract

In this section we derive the optimal financial contract for the $n = 2$ case considered in this paper. Bernanke & Gertler (1989) give details of the solution to general case in the context of their model, from which this draws heavily (adapted to the simper $n = 2$ case). First, we set up the Lagrangian:

$$\begin{aligned}\mathcal{L} = {} & \pi_L(pc^a + (1-p)c_L) + \pi_H c_H \\ & - \lambda_1\big[r(x(\omega) - NW^e) - \pi_L(\hat{q}k_L - p(c^a + \hat{q}\gamma) - (1-p)c_L) - \pi_H(\hat{q}k_H - c_H)\big] \\ & - \lambda_2\big[(1-p)(\hat{q}(k_H - k_L) + c_L) - c_H\big] \\ & + \lambda_3 c_L + \lambda_4 c^a + \lambda_5 p - \lambda_6(p-1)\end{aligned} \tag{33}$$

The KKT conditions are the following:

$$\frac{\partial \mathcal{L}}{\partial p} = \pi_L(c^a - c_L) - \lambda_1\pi_L(c^a + \hat{q}\gamma - c_L) + \lambda_2(\hat{q}(k_H - k_L) + c_L) + \lambda_5 - \lambda_6 = 0 \tag{34}$$

$$\frac{\partial \mathcal{L}}{\partial c^a} = \pi_L p - \lambda_1\pi_L p + \lambda_4 = 0 \tag{35}$$

$$\frac{\partial \mathcal{L}}{\partial c_L} = \pi_L(1-p) - \lambda_1\pi_L(1-p) - \lambda_2(1-p) + \lambda_3 = 0 \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial c_H} = \pi_H - \lambda_1\pi_H + \lambda_2 = 0 \tag{37}$$

$$\lambda_1\big[r(x(\omega) - NW^e) - \pi_L(\hat{q}k_L - p(c^a + \hat{q}\gamma) - (1-p)c_L) - \pi_H(\hat{q}k_H - c_H)\big] = 0 \tag{38}$$

$$\lambda_2\big[(1-p)(\hat{q}(k_H - k_L) + c_L) - c_H\big] = 0 \tag{39}$$

$$\lambda_3 c_L = 0 \tag{40}$$

$$\lambda_4 c^a = 0 \tag{41}$$

$$\lambda_5 p = 0 \tag{42}$$

$$\lambda_6(p-1) = 0 \tag{43}$$

$$\lambda_1 \geq 0 \tag{44}$$

$$\lambda_2 \geq 0 \tag{45}$$

$$\lambda_3 \geq 0 \tag{46}$$

$$\lambda_4 \geq 0 \tag{47}$$

$$\lambda_5 \geq 0 \tag{48}$$

$$\lambda_6 \geq 0 \tag{49}$$

Now, we firstly assume that $\lambda_1 < 1$ (and note that $\lambda_1 \geq 0$ from (44)). Observe that from (37) we obtain that $(1-\lambda_1)\pi_H + \lambda_2 = 0$. Since $\pi_H > 0$ by assumption and $\lambda_2 \geq 0$ by (45) this is a contradiction and we must have $\lambda_1 \geq 1$.

Claim 3. *If $\lambda_1 = 1$ then $p = 0$.*

Proof. Suppose $\lambda_1 = 1$. Then, from (35) and (37) we immediately obtain $\lambda_4 = 0$ and $\lambda_2 = 0$ respectively. Using the latter in (36) we also obtain that $\lambda_3 = 0$. Then, substituting all these into (34) we obtain:

$$-\pi_L(\hat{q}\gamma) + \lambda_5 - \lambda_6 = 0, \tag{50}$$

which immediately implies that $\lambda_5 > 0$, since $\lambda_6 \geq 0$ from (49) and $\pi_L(\hat{q}\gamma) > 0$. Thus, $p = 0$ by (42). $\square$

Now, we know that the constraint with multiplier λ_1 always binds, we can write

$$\pi_H c_H = \pi_L(\hat{q}k_L - p(c^a + \hat{q}\gamma) - (1-p)c_L) + \pi_H\hat{q}k_H - r(x(\omega) - NW^e), \tag{51}$$

and substituting this into the objective function we obtain:

$$\begin{aligned}
&\pi_L(pc^a + (1-p)c_L) + \pi_H c_H \\
&= \pi_L(pc^a + (1-p)c_L) + \pi_L(\hat{q}k_L - p(c^a + \hat{q}\gamma) - (1-p)c_L) + \pi_H\hat{q}k_H - r(x(\omega) - NW^e) \\
&= \hat{q}(\pi_L k_L + \pi_H k_H - p\gamma) - r(x(\omega) - NW^e).
\end{aligned} \tag{52}$$

Since the only remaining choice variable is p, the problem becomes to minimise expected agency costs: $\hat{q}p\gamma$.

Claim 4. *The optimal contract has $p = 0$ if and only if $r(x(\omega) - NW^e) \leq \hat{q}k_L$.*

Proof. Suppose $p = 0$. Then, from the constraint with multiplier λ_2, we have $c_H \geq \hat{q}(k_H - k_L) + c_L \geq \hat{q}(k_H - k_L)$. Substituting this into the first constraint, we obtain:

$$\begin{aligned}
r(x(\omega) - NW^e) &\leq \pi_L(\hat{q}k_L - c_L) + \pi_H(\hat{q}k_H - c_H) \\
&\leq \pi_L(\hat{q}k_L - c_L) + \pi_H(\hat{q}k_H - \hat{q}k_H + \hat{q}k_L) \\
&= (\pi_L + \pi_H)\hat{q}k_L - \pi_L c_L \\
&= \hat{q}k_L - \pi_L c_L
\end{aligned} \tag{53}$$

and so $\hat{q}k_L \geq r(x(\omega) - NW^e)$ as claimed. For necessity, suppose that $\hat{q}k_L \geq r(x(\omega) - NW^e)$ and consider that contract in which $c_i = \hat{q}k_L - r(x(\omega) - NW^e)$ for $i = L, H$, $p = 0$ and c^a is immaterial satisfies all constraints and minimises $\hat{q}p\gamma$ (which is bounded from below by 0). Given the remark above, this contract necessarily solves the problem. $\square$

From now on, we will assume that $\hat{q}k_L < r(x(\omega) - NW^e)$, so that $p > 0$ and therefore $\lambda_1 > 0$ by the previous two claims. The condition (35), $(1 - \lambda_1)\pi_L p + \lambda_4 = 0$ immediately implies that $\lambda_4 > 0$ and so $c^a = 0$. An analogous argument can be made from (37) to obtain that $\lambda_2 > 0$. This implies that $(1 - p)(\hat{q}(k_H - k_L) + c_L) = c_H$ by (39). We can also see that $c_L = 0$ since (36) implies that $\lambda_3 > 0$ which, in conjunction with (40) implies the result. Therefore, we can be more precise: $c_H = (1 - p)\hat{q}(k_H - k_L)$.

With all this, we can rewrite the binding first constraint as:

$$\begin{aligned} \pi_L[\hat{q}(k_L - p\gamma)] + \pi_H[\hat{q}k_H - (1-p)\hat{q}(k_H - k_L)] &= r(x(\omega) - NW^e) \\ \implies \quad p[\pi_H \hat{q}(k_H - k_L) - \pi_L \hat{q}\gamma] &= r(x(\omega) - NW^e) - \hat{q}k_L \\ \implies \quad p &= \frac{r(x(\omega) - NW^e) - \hat{q}k_L}{\pi_H \hat{q}(k_H - k_L) - \pi_L \hat{q}\gamma}. \end{aligned} \tag{54}$$

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